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LECTURER IN MATHEMATICS

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MATRICES

Matrix, a set of numbers arranged in rows and columns so as to form a rectangular array.

If there are m rows and n columns, the matrix is said to be an “ m by n ” matrix, written “ $m \times n$.”

For example, $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a 2×3 matrix. In a common notation, a capital letter denotes a matrix, and the corresponding small letter with a double subscript describes an element of the matrix. Thus, a_{ij} is the element in the i^{th} row and j^{th} column of the matrix A . If A is the 2×3 matrix shown above, then $a_{11} = 1$, $a_{12} = 2$, $a_{13} = 3$, $a_{21} = 4$, $a_{22} = 5$, and $a_{23} = 6$.

Types of Matrices

Row Matrix

A matrix having only one row is called a **row matrix**. Thus $A = [a_{ij}]_{m \times n}$ is a row matrix if $m = 1$.

So, a row matrix can be represented as $A = [a_{ij}]_{1 \times n}$. It is called so because it has only one row, and the order of a row matrix will hence be $1 \times n$.

Column Matrix

A matrix having only one column is called a **column matrix**. Thus, $A = [a_{ij}]_{m \times n}$ is a column matrix if $n = 1$. So, the value of a column matrix will be 1. Hence, the order is $m \times 1$.

Just like the row matrices had only one row, column matrices have only one column. Thus, the value of a column matrix will be 1. Hence, the order is $m \times 1$. The general form of a column matrix is given by $A = [a_{ij}]_{m \times 1}$. Other examples of a column matrix include:

Zero or Null Matrix

If all the elements are zero in a matrix, then it is called a zero matrix and generally denoted by O. Thus, $A = [a_{ij}]_{m \times n}$ is a zero-matrix if $a_{ij} = 0$ for all i and j;

Singleton Matrix

If there is only one element in a matrix, it is called a **singleton matrix**. Thus, $A = [a_{ij}]_{m \times n}$ is a singleton matrix if $m = n = 1$.

Horizontal Matrix

A matrix of order $m \times n$ is a **horizontal matrix** if $n > m$;

Vertical Matrix

A matrix of order $m \times n$ is a **vertical matrix** if $m > n$;

Square Matrix

If the number of rows and the number of columns in a matrix are equal, then it is called a **square matrix**.

Thus, $A = [a_{ij}]_{m \times n}$ is a square matrix if $m = n$;

Ex: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is a square matrix of order 3×3 .

Rectangular Matrix

A matrix which is not a square matrix is called a rectangular matrix.

Ex: $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ is a rectangular matrix of order 2×3 .

Diagonal Matrix

If all the elements, except the principal diagonal, in a square matrix, are zero, it is called a **diagonal matrix**. Thus, a square matrix $A = [a_{ij}]$ is a diagonal matrix if $a_{ij} = 0$, when $i \neq j$.

The special thing is that all the non-diagonal elements of this matrix are zero. That means only the diagonal has non-zero elements. There are two important things to note here, which are as follows:

- (i) A diagonal matrix is always a square matrix.
- (ii) The diagonal elements are characterized by this general form: a_{ij} where $i = j$. This means that a matrix can have only one diagonal.

Scalar Matrix

If all the elements in the diagonal of a diagonal matrix are equal, it is called a **scalar matrix**.

Unit Matrix or Identity Matrix

If all the elements of a principal diagonal in a diagonal matrix are 1, it is called a **unit matrix**.

A unit matrix of order n is denoted by I_n . Thus, a square matrix $A = [a_{ij}]_{m \times n}$ is an [identity matrix](#) if

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Conclusions:

- All identity matrices are scalar matrices
- All scalar matrices are diagonal matrices
- All diagonal matrices are square matrices

It should be noted that the converse of the above statements is not true for any of the cases.

Equal Matrices

Equal matrices are those matrices which are equal in terms of their elements. The conditions for matrix equality are discussed below.

Equality of Matrices Conditions

Two matrices A and B are said to be equal if they are of the same order and their corresponding elements are equal, i.e. two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{r \times s}$ are equal if:

- (a) $m = r$, i.e., the number of rows in A = the number of rows in B.
- (b) $n = s$, i.e. the number of columns in A = the number of columns in B
- (c) $a_{ij} = b_{ij}$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, i.e. the corresponding elements are equal;

Triangular Matrix

A square matrix is said to be a **triangular matrix** if the elements above or below the principal diagonal are zero, and there are of two types:

Upper Triangular Matrix

A square matrix $[a_{ij}]$ is called an **upper triangular matrix**, if $a_{ij} = 0$, when $i > j$.

Lower Triangular Matrix

A square matrix is called a **lower triangular matrix**, if $a_{ij} = 0$ when $i < j$.

Singular Matrix and Non-Singular Matrix

Matrix A is said to be a **singular matrix** if it's determinant $|A| = 0$; otherwise, a **non-singular matrix**, i.e. if for $\det |A| = 0$, it is singular matrix and for $\det |A| \neq 0$, it is non-singular.

Symmetric and Skew Symmetric Matrices

Symmetric matrix: A square matrix $A = [a_{ij}]$ is called a symmetric matrix if $a_{ij} = a_{ji}$, for all i, j values;

Note: A is symmetric if $A' = A$ (where 'A' is the transpose of the matrix)

Skew-Symmetric Matrix: A square matrix $A = [a_{ij}]$ is a skew-symmetric matrix if $a_{ij} = -a_{ji}$, for all values of i,j.

Note: A square matrix A is a skew-symmetric matrix $A' = -A$

Some Important Conclusions on Symmetric and Skew-Symmetric Matrices

- If A is any square matrix, then $A + A'$ is a symmetric matrix and $A - A'$ is a skew-symmetric matrix.
- Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}(B + C)$$

where B is symmetric, and C is a skew-symmetric matrix.

If A and B are symmetric matrices, then AB is symmetric $AB = BA$, i.e., A & B commute.

- The matrix $B'AB$ is symmetric or skew-symmetric in correspondence if A is symmetric or skew-symmetric.
- All positive integral powers of a symmetric matrix are symmetric.
- Positive odd integral powers of a skew-symmetric matrix are skew-symmetric, and positive even integral powers of a skew-symmetric matrix are symmetric.

Hermitian and Skew-Hermitian Matrices

A square matrix $A = [a_{ij}]$ is said to be a Hermitian matrix if $a_{ij} = \overline{a_{ji}}$ for all i, j. i.e $A = A^{\theta}$

Important Notes:

- If A is a Hermitian matrix, then $a_{ii} = \overline{a_{ii}} \Rightarrow a_{ii}$ is real for all i, thus every diagonal element of a Hermitian Matrix must be real.

- If a Hermitian matrix over the set of real numbers is actually a real symmetric matrix; and A a square matrix, $A = [a_{ij}]$ is said to be a skew-Hermitian if $a_{ij} = -\overline{a_{ji}}$ for all i,j

i.e., $A = -A^{\theta}$

- If A is a skew-Hermitian matrix then $a_{ii} = -\overline{a_{ii}} \Rightarrow a_{ii} + a_{ii} = 0$

i.e., a_{ii} must be purely imaginary or zero.

- A skew-Hermitian matrix over the set of real numbers is actually a real skew-symmetric matrix.

Multiplication of a Matrix by a Scalar

Matrix scalar multiplication is multiplying a matrix by a scalar. A scalar is a real number whereas a matrix is a rectangular array of numbers. When we deal with matrices, we come across two types of multiplications:

- Multiplying a matrix by another matrix and is called "matrix multiplication"
- Multiplying a matrix by a scalar (a number) and is called "matrix scalar multiplication".

The **matrix scalar multiplication** is the process of multiplying a matrix by a scalar. Let 'A' be a matrix and 'k' be a scalar (real number). Then kA is the result of the matrix scalar multiplication.

Thus, matrix scalar multiplication is mathematically defined as follows:

"If $A = [a_{ij}]_{m \times n}$ and k is a scalar then $kA = k [a_{ij}]_{m \times n} = [ka_{ij}]_{m \times n}$ "

i.e., the element in i^{th} row and j^{th} column of kA is obtained by multiplying the corresponding element of A by 'k'.

Properties of Matrix Scalar Multiplication

If A and B are matrices of the same order; and k, a, and b are scalars then:

- A and kA have the same order.

- Matrix scalar multiplication is commutative. i.e., $kA = Ak$.
- Scalar multiplication of matrices is associative. i.e., $(ab)A = a(bA)$.
- The distributive property works for the matrix scalar multiplication as follows:
 $k(A + B) = kA + kB$
 $A(a + b) = Aa + Ab$ (or) $aA + bA$
- The product of any scalar and a zero matrix is the zero matrix itself.
- A is symmetric $\Rightarrow KA$ is symmetric.
- A is skew-symmetric $\Rightarrow KA$ is skew-symmetric.
- $(KA)^T = KA^T$
- $KA = 0$ if and only if $A = 0$ if $K \neq 0$.

Sum of Matrices or Matrix Addition

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ be two matrices. The matrix $C = [c_{ij}]_{m \times n}$ where $c_{ij} = a_{ij} + b_{ij}$ is called the sum of the matrices A and B . The sum of A and B is denoted by $A + B$.

Thus $[a_{ij}]_{m \times n} + [b_{ij}]_{m \times n} = [a_{ij} + b_{ij}]_{m \times n}$.

The difference (subtraction) of two matrices

If A, B are two matrices of the same type (order), then $A + (-B)$ is taken as $A - B$.

Thus if $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ then $A - B = [a_{ij}]_{m \times n} - [b_{ij}]_{m \times n} = [a_{ij} - b_{ij}]_{m \times n}$.

Matrix Multiplication

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$. The matrix $C = [c_{ij}]_{m \times p}$ where $c_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$ is called the product of the matrices A and B in that order.

In the product AB , the matrix A is called the pre-factor and B the post-factor.

If the number of columns of A is equal to the number of rows in B , then the matrices are said to be conformable for multiplication in that order.

Note: If A, B are matrices, then

- AB may be possible and BA may not be possible.
- BA may be possible and AB may not be possible.
- AB, BA may both be possible.
- Even if AB and BA are possible, they may not be equal.
- AB and BA are possible when A, B are each of order $n \times n$. In this also they may not be equal.
- Multiplication of matrices is not commutative. If A and B are matrices such that $AB = BA$, then we say that A and B commute.
- If A, B are conformable for multiplication and k, l are scalars, the $(kA)B = k(AB) = A(kB)$ and $(kA)(lB) = (kl)(AB)$.
- Matrix multiplication is associative. i.e if A, B, C are matrices then $(AB)C = A(BC)$.
- Multiplication of matrices is distributive w.r.t. addition of matrices.
i.e., $A(B+C) = AB + AC$.

Trace of a Matrix

The sum of the diagonal elements in a square matrix A is called the trace of matrix A , and which is denoted by $\text{tr}(A)$;

$$\text{tr}(A) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties

If A and B are square matrices of order n and λ is any scalar, then

- $\text{tr}(\lambda A) = \lambda \text{tr} A$
- $\text{tr}(A+B) = \text{tr} A + \text{tr} B$
- $\text{tr}(AB) = \text{tr}(BA)$.

Idempotent Matrix

If A is a square matrix such that $A^2 = A$, then A is called idempotent matrix.

Nilpotent Matrix

If A is a square matrix such that $A^m = O$ where m is a positive integer, then A is called “nilpotent”. If m is least positive integer such that $A^m = O$, then A is called nilpotent of index m.

Involutory Matrix

If A is a square matrix such that $A^2 = I$, then A is called involutory.

Determinant of a Matrix

The determinant of a matrix is a number that is specially defined only for square matrices. Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations. Determinants also have wide applications in engineering, science, economics and social science as well. Let’s now study about the determinant of a matrix.

Symbol

The determinant of a matrix is represented by two vertical lines or simply by writing det and writing the matrix name. It is also denoted by $|A|$ or $\det A$ or Δ .

Minors and Cofactors of a Square Matrix

Let $A = [a_{ij}]_{m \times n}$ be a square matrix, when from A the elements of its i^{th} row and j^{th} column are deleted, the determinant of $(n - 1)$ rowed matrix M_{ij} is called the minor of a_{ij} of A and is denoted by $|M_{ij}|$. The signed minor $(-1)^{i+j} |M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij} . Thus $|M_{ij}|$ for all I and j are called minors of A and A_{ij} for all I and j are called cofactors of A.

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then

- i) the minor of a_{12} is $|M_{12}| = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, etc.,
- ii) the cofactor a_{12} is $A_{12} = (-1)^{1+2} |M_{12}| = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{32} \end{vmatrix}$
- iii) $|A| = a_{11} |M_{11}| - a_{12} |M_{12}| + a_{13} |M_{13}| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}$.

Calculating the Determinant

To find a Determinant of a matrix, for every square matrix $[A]_{n \times n}$ there exists a determinant to the matrix such that it represents a unique value given by applying some determinant finding techniques.

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } \Delta = \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{12} \times a_{21}$$

Note down the difference between the representation of a matrix and a determinant. In the case of a matrix, we enclose the value in a square bracket whereas in case of a determinant we enclose it in between two lines.

Properties of Determinants

- If rows and columns in square matrix A be interchanged, then the determinant of the resulting matrix will be $|A|$. i.e. $|A^T| = |A|$.
- If every element of a column of a square matrix A is zero, then $|A| = 0$.
- If interchange of two rows (columns) in a square matrix A is made, then the determinate of the resulting matrix will be $-|A|$.
- If two rows (columns) of a square matrix A are identical, then $|A| = 0$.
- If every element of a row (column) of a square matrix A be multiplied by a number λ , then the determinant of the resulting matrix will be $\lambda |A|$.
- If A be a square matrix of order n , then $|\lambda A| = \lambda^n |A|$.
- If A, B are square matrices conformable for multiplication, then $|AB| = |A| |B|$.
- If A is a skew symmetric matrix of order n (an odd integer), then $|A| = 0$.
- If A, B are square matrices each of order n such that $|AB| = 0$ then $|A| = 0$ or $|B| = 0$.
- If $|A| = 0$ then $|\text{adj } A| = 0$.
- If A is a square matrix of order n , then $|\text{adj } A| = |A|^{n-1}$.
- If A is a symmetric matrix then $\text{adj } A$ is also symmetric.
- If A is a square matrix then $\text{adj } A' = (\text{adj } A)'$.

Orthogonal Matrix:

A square matrix A is said to be orthogonal if $A^T A = I$.

Rank of a Matrix:

If A is a null matrix, then the rank of A is defined as zero.

If A is not a null matrix, then the rank of the matrix A is the positive integer, if it possesses the following two properties.

- Each minor of order $n+1$ of the matrix A vanishes.
- There exists at least one minor of order n of the matrix A which is not zero.

Note:

- Rank of A is denoted by $\rho(A)$ and is unique.
- Every matrix will have a rank.
- If A is a matrix of order $m \times n$ then $\rho(A) \leq m$ or n (smaller of the two).
- If $\rho(A) = n$, then every minor of order $n+1$, $n+2$, etc. is zero.
- If A is a matrix of order $n \times n$. A is non-singular if and only if $\rho(A) = n$.
- Rank of $I_n = n$.
- Elementary transformations on a matrix do not alter the rank of the matrix.
- The rank of a matrix does not change by pre-multiplication or post-multiplication with a non-singular matrix.
- The rank of the transpose of a matrix is equal to the rank of the original matrix.
i.e $\rho(A) = \rho(A')$.

Normal form:

Every non-zero matrix can be reduced to the form $I_r, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, [I_r, 0], \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ called its normal form. Hence the rank of the given matrix is 'r'.

Echelon form:

A matrix is said to be in Echelon form if it has the following properties.

- Zero matrix, if any, must follow non-zero rows.
- The first non-zero element in each non-zero row must be 1.
- The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Note:

- The rank of a product of two matrices cannot exceed the rank of either matrix.
i.e $\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$.
- Rank of the sum of two matrices cannot exceed the sum of their ranks.
i.e $\rho(A + B) \leq \rho(A) + \rho(B)$.

- If A, B are two n-rowed square matrices, then $\rho(A+B) \geq \rho(A) + \rho(B) - n$.

The equation $b = a_1x_1 + a_2x_2 + \dots + a_nx_n$ -----(1) expressing b in terms of the variables $x_1, x_2, \dots, x_n$ and the constants $a_1, a_2, \dots, a_n$ is called a linear equation.

Characteristic Vector

Any non-zero vector X is said to be a characteristic vector of a square matrix A if there exists a scalar λ such that $AX = \lambda X$

Here A can be $n \times n$ matrix and X can be $n \times 1$ matrix.

Then λ is said to be a characteristic value of the matrix A corresponding to the characteristic vector X. Also, X is said to be characteristic vector corresponding to the characteristic value λ of the matrix A. Characteristic vectors are sometimes called Proper or latent or Eigen vectors.

Characteristic polynomial

Let $A = [a_{ij}]_{n \times n}$ and λ any indeterminate scalar. The matrix $A - \lambda I$ is called the characteristic matrix of A, where I is the unit matrix of order n. Also $|A - \lambda I|$, a polynomial in λ of degree n, is called the characteristic polynomial of A.

Characteristic Equation

The equation $|A - \lambda I| = 0$ is called the characteristic equation of A and λ is a root of the characteristic equation of A.

Ex: $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$. The characteristic eqn is $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = -2, 5$

- A scalar λ is a characteristic root of a square matrix A if and only if $|A - \lambda I| = 0$.
- The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.
- The characteristic roots of any diagonal matrix are same as its elements in the diagonal.
- The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.
- The square matrix A and A^T have the same characteristic values.
- If 0 is a characteristic root of a matrix if and only if the matrix is singular.
- λ is a characteristic root of a non-singular matrix then $\lambda \neq 0$.
- At least one characteristic root of every singular matrix is zero.

- If λ is a characteristic root of the matrix A and k is a scalar then $k + \lambda$ is a characteristic root of the matrix $A + kI$.
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of a n -rowed square matrix A and k is a scalar then $k\lambda_1, k\lambda_2, \dots, k\lambda_n$ are the characteristic values of kA .
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of a n -rowed square matrix A and k is a scalar then $\lambda_1 - k, \lambda_2 - k, \dots, \lambda_n - k$ are the characteristic values of $A - kI$.
- If λ is an eigen value of the matrix A , then $k + \lambda$ is an eigen value of $A + kI$.
- If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the characteristic values of a n -rowed square matrix A then $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ are the characteristic roots of A^2 .
- If the matrix a is non-singular, then the eigen values of A^{-1} are the reciprocal of the eigen value of A .

Cayley – Hamilton theorem

Every square matrix satisfies its characteristic equation.
